e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Topic:Homomorphism of group

HOMOMORPHISM

Definition and Examples

Definition

A map ϕ of a group G into a group G' is a **homomorphism** if the homomorphism property $\phi(ab) = \phi(a)\phi(b)$ holds for all $a, b \in G$.

Remark.

For any groups G and G', there is always at least one homomorphism $\phi : g \rightarrow G'$ namely the *trivial homomorphism* defined by $\phi(g) = e'$ for all $g \in G$, where e' is the identity element of G'.

Example

Let $\phi : G \rightarrow G'$ be a group homomorphism of G onto G'. Then G' will be abelian if G is abelian. To see this, let $a', b' \in G'$. Since ϕ is onto, there exists $a, b \in G$ such that $\phi(a) = a'$ and $\phi(b) = b'$. Then $a'b' = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = b'a'$, where the third equality is due to the fact that G is abelian. This shows that G' is abelian. Thus, this example illustrates how one can get information about G' from a given information about G via a homomorphism $\phi : G \rightarrow G'$.

Example

Let $GL(n, \mathbb{R})$ be the multiplicative group of all invertible $n \times n$ matrices. Then $\phi : GL(n, \mathbb{R}) \to \mathbb{R}^*$ defined by $\phi(A) = detA$, the *determinant* of A, for all $A \in GL(n, \mathbb{R})$ is a homomorphism, since $det(AB) = det(A)det(B)$ and since $det(A) \neq 0$ for any invertible $n \times n$ matrix A.

Problem

Determine whether the given map ϕ is a homomorphism.

(a) Let $\phi : \mathbb{Z} \to \mathbb{R}$ under addition be given by $\phi(n) = n$.
(b) Let $\phi : \mathbb{R} \to \mathbb{Z}$ under addition be given by $\phi(x) =$ the greatest integer $\leq x$.

(c) Let $\phi : \mathbb{R}^* \to \mathbb{R}^*$ under multiplication be given by $\phi(x) = |x|$.

(d) Let $\phi : \mathbb{R} \to \mathbb{R}^*$ where $\mathbb{R}$ is additive and $\mathbb{R}^*$ is multiplicative, be given by $\phi(x) = 2^x$.

Solution.

(a) It is a homomorphism, because $\phi(m + n) = m + n = \phi(m) + \phi(n)$.

(b) It is not a homomorphism, because $\phi(2.6 + 1.6) = \phi(4.2) = 4$ but $\phi(2.6) + \phi(1.6) = 2 + 1 = 3$.

(c) It is a homomorphism, because $\phi(xy) = |xy| = |x||y| = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}^*$

(d) It is a homomorphism, because $\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}^*$. ■

Problem

Let $M_n(\mathbb{R})$ be the additive group of all $n \times n$ matrices with real entries, and let $\mathbb{R}$ be the additive group of real numbers. Determine whether the given map ϕ is a homomorphism.

(a) Let $\phi : M_n(\mathbb{R}) \to \mathbb{R}$ be given by $\phi(A) = det(A)$, the determinant of $A \in M_n(\mathbb{R})$.

(b) Let $\phi : M_n(\mathbb{R}) \to \mathbb{R}$ be given by $\phi(A) = tr(A)$, the trace of $A \in M_n(\mathbb{R})$.

*(The **trace** of A, $tr(A)$ is the sum of the elements on the main diagonal of A.)*

Solution.

(a) No, it is not a homomorphism. Let $n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so that $A + B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. We see that $\phi(A + B) = det(A + B) = 4 - 1 = 3$, but $\phi(A) + \phi(B) = det(A) + det(B) = 1 + 0 = 1$.

(b) Yes, it is a homomorphism. Let $A = (a_{ij})$ and $B = (b_{ij})$ where the element with subscript ij is in the i^{th} row and j^{th} column. Then $\phi(A+B) = tr(A+B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = tr(A) + tr(B) = \phi(A) + \phi(B)$. ■