e content for students of patliputra university

B. Sc. (Honrs) Part 2paper

Subject:Mathematics

Topic:Transpose &Inverse of a matrix

## Transpose of a matrix

*Definition.* Given a matrix $A$, the **transpose** of $A$, denoted $A^T$, is the matrix whose rows are columns of $A$ (and whose columns are rows of $A$). That is, if $A = (a_{ij})$ then $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$.

*Examples.* $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}^T = (7, 8, 9), \qquad \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}^T = \begin{pmatrix} 4 & 7 \\ 7 & 0 \end{pmatrix}.$$

# properties of transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \ldots A_k)^T = A_k^T \ldots A_2^T A_1^T$
- $(A^{-1})^T = (A^T)^{-1}$

# Symmetric &Skew-symmetric matri x

DEFINITION. (Symmetric matrix) A real matrix $A$ is called *symmetric* if $A^t = A$. In other words $A$ is square ($n \times n$ say) and $a_{ji} = a_{ij}$ for all $1 \leq i \leq n$, $1 \leq j \leq n$. Hence

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

is a general $2 \times 2$ symmetric matrix.

DEFINITION. Skew–symmetric matrix) A real matrix $A$ is called *skew–symmetric* if $A^t = -A$. In other words $A$ is square ($n \times n$ say) and $a_{ji} = -a_{ij}$ for all $1 \leq i \leq n$, $1 \leq j \leq n$.

REMARK. Taking $i = j$ in the definition of skew–symmetric matrix gives $a_{ii} = -a_{ii}$ and so $a_{ii} = 0$. Hence

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

is a general $2 \times 2$ skew–symmetric matrix.

# Identity matrix

**Definition** The *identity matrix*, denoted $I_n$, is the $n$ x $n$ diagonal matrix with all ones on the diagonal.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $A$ is an $m$ x $n$ matrix, then

$$I_m A = A \quad \text{and} \quad A I_n = A.$$

If $A$ is a square matrix, then

$$IA = A = AI.$$

# Inverse of a martrix

Given a square matrix $A$, the *inverse of* $A$, denoted $A^{-1}$, is defined to be the matrix such that

$$AA^{-1} = A^{-1}A = I$$

Note that inverses are only defined for square matrices

**Note** Not all matrices have inverses.

If $A$ has an inverse, it is called *invertible*.

If $A$ is not invertible it is called *singular*.

**Theorem.** *If $A$ is invertible, then its inverse is unique.*

*Proof.* Assume $A$ is invertible. Suppose, by way of contradiction, tha
inverse of $A$ is not unique, i.e., let $B$ and $C$ be two distinct inverses (
Then, by def'n of inverse, we have

$$BA = I = AB \quad (1)$$
$$\text{and } CA = I = AC. \quad (2)$$

It follows that

$$\begin{aligned} B &= BI && \text{by def'n of identity matrix} \\ &= B(AC) && \text{by (2) above} \\ &= (BA)C && \text{by associativity of matrix mult.} \\ &= IC && \text{by (1) above} \\ &= C. && \text{by def'n of identity matrix} \end{aligned}$$

Thus, $B = C$, which contradicts the previous assumption that $B \neq$
$\Rightarrow\Leftarrow$ So it must be that case that the inverse of $A$ is unique.

**Inverse of a $2 \times 2$ matrix:** Consider the special case where $A$ is a $2 \times 2$ matrix with $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. If $ad - bc \neq 0$, then $A$ is invertible and its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

**Example** For $A = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix}$, we have

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix}.$$

We can easily check that

$$AA^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**Example:** Find the inverse of the matrix $A = \begin{bmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+3R_1} \left[\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & -3 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-R_2]{-R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{2}{3} & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

Thus, $A$ is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ 1 & \frac{2}{3} & -1 \\ 2 & 1 & -1 \end{bmatrix}.$$

**Theorem 11** *Given two invertible matrices $A$ and $B$*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

**Proof:** Let $A$ and $B$ be invertible matricies and let $C = AB$, so $C^{-1} = (AB)^{-1}$.

Consider $C = AB$.
Multiply both sides on the left by $A^{-1}$:

$$A^{-1}C = A^{-1}AB = B.$$

Multiply both sides on the left by $B^{-1}$.

$$B^{-1}A^{-1}C = B^{-1}B = I.$$

So, $B^{-1}A^{-1}$ is the matrix you need to multiply $C$ by to get the identity.

Thus, by the definition of inverse

$$B^{-1}A^{-1} = C^{-1} = (AB)^{-1}.$$

**Example** .

1. Find $A^{-1}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix}$$

Augment with $I$ and row reduce:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right) \quad R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right) \quad R_3 \to -\tfrac{1}{2}R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right) \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -13/2 & 3/2 & 3/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right) \quad R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -15/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right)$$

So

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}$$