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B. Sc. (Honrs) Part 1 paper 1

Subject:Mathematics

Title/Heading:Groups:Algebra of sets, De Morgan's laws

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Algebra of sets &De Morgan's laws

Example. (**Distributivity**) Let A, B, and C be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If X and Y are sets, $X = Y$ if and only if for all x, $x \in X$ if and only if $x \in Y$.
First, I'll give a formal proof, written as a series of double implications:

$$\begin{array}{rcll}
x \in A \cap (B \cup C) & \leftrightarrow & x \in A \wedge x \in (B \cup C) & \text{Definition of } \cap \\
& \leftrightarrow & x \in A \wedge (x \in B \vee x \in C) & \text{Definition of } \cup \\
& \leftrightarrow & (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) & \text{Distributivity of } \wedge \text{ over} \\
& \leftrightarrow & (x \in A \cap B) \vee (x \in A \cap C) & \text{Definition of } \cap \\
& \leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{Definition of } \cup
\end{array}$$

I've shown that

$$x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C).$$

By definition of set equality, this proves that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □

Example. (DeMorgan's Law) Let A and B be sets. Prove that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Let x be an arbitrary element of the universe.

$$\begin{array}{rcll}
x \in \overline{A \cup B} & \leftrightarrow & x \notin A \cup B & \text{Definition of complement} \\
& \leftrightarrow & \neg(x \in A \cup B) & \text{Definition of } \notin \\
& \leftrightarrow & \neg(x \in A \vee x \in B) & \text{Definition of } \cup \\
& \leftrightarrow & \neg(x \in A) \wedge \neg(x \in B) & \text{DeMorgan's law} \\
& \leftrightarrow & (x \notin A) \wedge (x \notin B) & \text{Definition of } \notin \\
& \leftrightarrow & (x \in \overline{A}) \wedge (x \in \overline{B}) & \text{Definition of complement} \\
& \leftrightarrow & x \in \overline{A} \cap \overline{B} & \text{Definition of } \cap
\end{array}$$

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

Example. Let A and B be sets. Prove that $A \cap B \subset A$.

This example will show how you prove a *subset relationship*.

By definition, if X and Y are sets, $X \subset Y$ if and only if for all x, if $x \in X$, then $x \in Y$.

Take an arbitrary element x. Suppose $x \in A \cap B$ (conditional proof). I want to show that $x \in A$.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, by definition of intersection. But $x \in A$ and $x \in B$ implies $x \in A$ (decomposing a conjunction), and this is what I wanted to show. Therefore, $A \cap B \subset A$.

By the way, you usually don't write the logic out in such gory detail. The proof above could be shortened to the following.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, so in particular $x \in A$. Therefore, $A \cap B \subset A$.

The "in particular" substitutes for decomposing the conjunction. $\square$

Example. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

$$
\begin{gathered}
x \in (A - B) \cup (B - A) \leftrightarrow \\
x \in (A - B) \vee x \in (B - A) \leftrightarrow \\
(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \leftrightarrow \\
[x \in A \vee (x \in B \wedge x \notin A)] \wedge [x \notin B \vee (x \in B \wedge x \notin A)] \leftrightarrow \\
(x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)] \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin A) \leftrightarrow \\
(x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A) \leftrightarrow \\
(x \in A \vee x \in B) \wedge (\neg x \in B \vee \neg x \in A) \leftrightarrow \\
(x \in A \vee x \in B) \wedge \neg(x \in B \wedge x \in A) \leftrightarrow \\
(x \in A \cup B) \wedge \neg(x \in A \cap B) \leftrightarrow
\end{gathered}
$$

$$x \in (A \cup B) - (A \cap B)$$

Therefore, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. □