B.Sc. (Math) part III
paper - VI

Topic :- Automorphism of Groups

Def :- An isomorphism of a group G onto itself is called automorphism of the group.

Theorem ① The set of all automorphisms of a group form a group with respect to composite of functions as the composition.

proof :- Let $A(G)$ be the collection of all automorphism of a group G then $A(G) =$
$\{f : f \text{ is an automorphism } G\}$
We need to prove that $A(G)$ is a group w.r.t. composite of function as composition.

① Closure property

Let $f, g \in A(G)$ Then f, g are one-one mapping of G onto itself. Therefore gf is also a one-one mapping of G onto itself

Let a, b be any two elements $\in G$ Then we have

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$(gf)(ab) = g\{f(ab)\} = g[f(a)f(b)]$ $\because f$ is isomorphism

$= g\{f(a)\}\{g(f(b))\}$

$\because g$ is an isomorphism

$= [g(f(a))] \; gf(b)]$

$\therefore gf$ is also an automorphism

Thus $A(G)$ is called w.r.to composite composition.

<u>Associativity</u>

It is known that composition of any arbitrary mapping f, g, h are associative. Hence composite of automorphism is also associative.

<u>Existence of identity</u>

i on G is the identity function also an automorphism of G clearly i is one-one onto and $\forall a, b \in G$ then

$i(ab) = ab = i(a)\, i(b)$

Thus $i \in A(G)$ and if $f \in A(G)$ we have $if = f = fi$

<u>Existance of inverse</u>

suppose $f \in A(G)$. since f is a one-one mapping of G onto itself therefore f^{-1} exits

$a \to a$ is also a one-one mapping of G onto itself: we show that f^{-1} is also automorphism of G

Let $a, b \in G$ then there exists $a', b' \in G$ such that $f^{-1}(a) = a'$

$\Leftrightarrow f(a') = a$

$f^{-1}(b) = b' \Leftrightarrow f(b') = b$

we have $f^{-1}(ab) = f^{-1}[f(a') f(b')]$

$\therefore f^{-1}$ is an automorphism of G $a \to a$ Thus $f \in A(G) \Rightarrow f^{-1} \in A(G)$

So each element of $A(G)$ posses its inverse

Hence $A(G)$ is a group with respect to composite composition