e-content for students

B. Sc.(honours) Part 1paper 2

Subject:Mathematics

Topic:Angle between two lines

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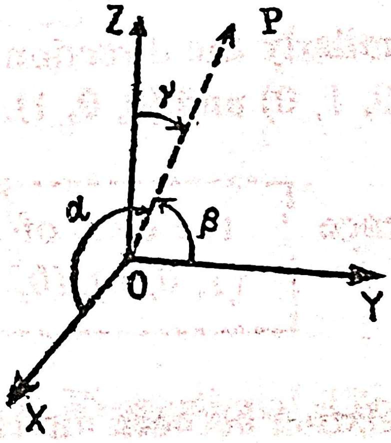
Direction Cosines of a Line

When a line in space is taken in a definite sense from one extreme to the other, the line is said to be directed.



For a directed line OP passing through the origin the angles α, β and γ, formed by OP with the x, y and z axes respectively, are called the direction angles of OP, and the cosines of these angles, that is, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines (d. cs., in short) of OP.

The direction cosines of a line are generally denoted by l, m, n.

Then $$\boxed{l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma.}$$

Relation between direction cosines

If a directed line through the origin makes angles α, β, γ with the x, y, z axes respectively, to prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

or $$l^2 + m^2 + n^2 = 1,$$

where l, m, n are the direction cosines of the directed line.

Let OP be the directed line through the origin making angles α, β, γ with the x, y, z axes respectively.

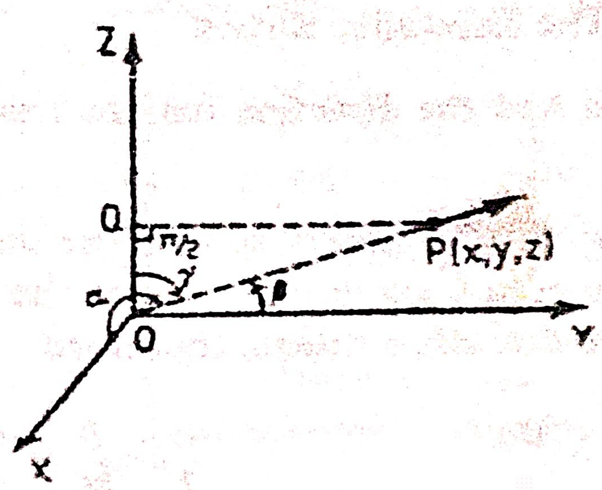
Consider now any point $P(x, y, z)$ on the directed line.

Let $OP=r$.

From P draw PQ perpendicular to the z-axis.

Then $\qquad OQ=z$

and $\qquad \angle QOP=\gamma$.



From right angled triangle QOP,

$$\cos\gamma=\frac{OQ}{OP}=\frac{z}{r} \quad \text{or} \quad z=r\cos\gamma.$$

Similarly drawing perpendiculars from P on x and y axes, we get

$$x=r\cos\alpha \quad \text{and} \quad y=r\cos\beta.$$

Now $\qquad OP=r$

or $\qquad \sqrt{x^2+y^2+z^2}=r \quad$ or $\quad x^2+y^2+z^2=r^2$

or $\qquad r^2\cos^2\alpha+r^2\cos^2\beta+r^2\cos^2\gamma=r^2$

or $\qquad r^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)=r^2$

or $\qquad \boxed{\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.}$

If $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$,

then $\qquad \boxed{l^2+m^2+n^2=1.}$

Direction Ratios or Direction Number

Any set of numbers a, b, c that are proportional to the direction cosines of a line are called *direction numbers* or *direction ratios* of the line. They are written in the form $[a, b, c]$ or (a, b, c) or a, b, c and will be referred to as the *direction* of the line.

To find the direction Cosines from the direction ratios of a line

Let l, m, n and a, b, c be the direction cosines and the direction ratios of the given line. Since the direction cosines are proportional to the direction ratios, therefore

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=k \text{ (say)}$$

or $$a=kl,\ b=km,\ c=kn, \qquad \ldots (1)$$

where k is called the constant of proportionality.

Squaring and adding these equations, we get

$$a^2+b^2+c^2=k^2(l^2+m^2+n^2).$$

But $$l^2+m^2+n^2=1.$$

$\therefore \quad a^2+b^2+c^2=k^2; \qquad$ or $\qquad k=\pm\sqrt{a^2+b^2+c^2}.$

From (1), $$l=\frac{a}{k}=\frac{a}{\pm\sqrt{a^2+b^2+c^2}},$$

$$m=\frac{b}{k}=\frac{b}{\pm\sqrt{a^2+b^2+c^2}},$$

$$n=\frac{c}{k}=\frac{c}{\pm\sqrt{a^2+b^2+c^2}},$$

where the sign of radical is either positive throughout or negative throughout, depending on which of the two possible directions of the line is desired.

To find the direction cosines and the direction ratios of the line-segment joining two given points (x_1, y_1, z_1) and (x_2, y_2, z_2).

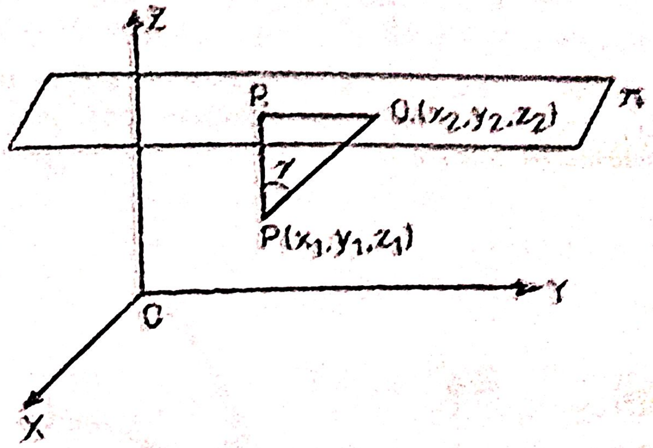
Let the given points (x_1, y_1, z_1) and (x_2, y_2, z_2) be respectively P and Q. Join P and Q. From P draw PR perpendicular to the plane π, passing through Q and parallel to XOY plane. Join R-Q. As $PR \perp$ plane π, it is perpendicular to every line in the plane π and hence to RQ. Clearly $PR = (z$ co-ordinate of $R)$

$-(z$ co-ordinate of $P)$

$= (z$ co-ordinate of $Q) - z_1$,

for R, Q are both on the plane $\pi \parallel$ plane XOY

$= z_2 - z_1$.



Let the d. cs. of PQ be l, m, n. Then $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

Now $\angle RPQ = \gamma$, as $PR \parallel OZ$.

From right angled $\triangle PRQ$,

$$\cos\gamma = \frac{PR}{PQ}; \text{ or } n = \frac{z_2 - z_1}{PQ}; \text{ or } PQ = \frac{z_2 - z_1}{n}.$$

Similarly $PQ = \dfrac{x_2 - x_1}{l}$ and $PQ = \dfrac{y_2 - y_1}{m}$.

Hence
$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ.$$

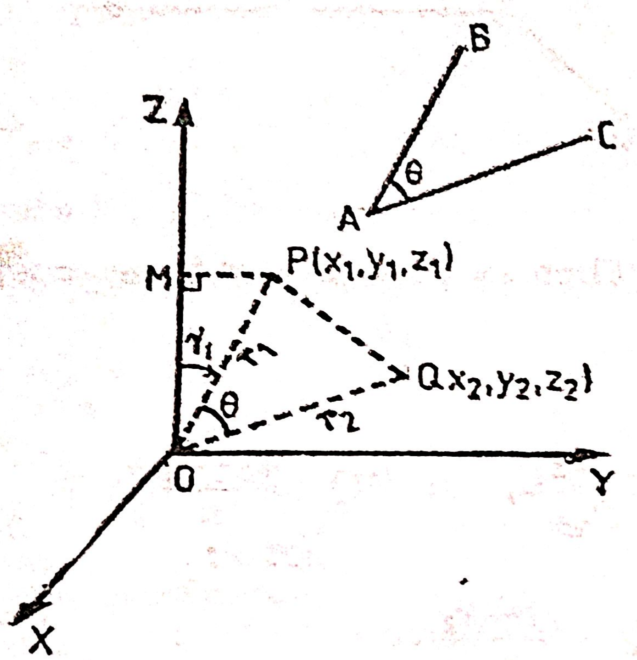
Thus (i) the direction cosines of PQ are

$$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$$

and (ii) the direction ratios of PQ are

$x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$.

To find the angle between two lines whose direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are given.



Let the given lines in space be AB and AC, whose direction cosines are respectively (l_1, m_1, n_1) and (l_2, m_2, n_2).

In order to find the angle θ between AB and AC, consider the radius vectors OP and OQ drawn from the origin and parallel to AB and AC respectively. Let P and Q be respectively (x_1, y_1, z_1) and (x_2, y_2, z_2) Obviously OP and OQ have the same directions as AB and AC, and hence the angle POQ is θ. Join P and Q. Let $OP=r_1$ and $OQ=r_2$.

From P draw PM perpendicular to the axis of z. Then $OM=z_1$.

Let
$$\angle MOP=\gamma_1.$$

From right angled triangle OMP,

$$\cos\gamma_1=\frac{OM}{OP}; \text{ or } n_1=\frac{z_1}{r_1}; \text{ or } z_1=n_1r_1.$$

Similarly $\qquad x_1=l_1r_1,\ y_1=m_1r_1$

and $\qquad x_2=l_2r_2,\ y_2=m_2r_2,\ z_2=n_2r_2.$

We have also $l_1^2+m_1^2+n_1^2=1$, $l_2^2+m_2^2+n_2^2=1$,

$$x_1^2+y_1^2+z_1^2=r_1^2 \text{ and } x_2^2+y_2^2+z_2^2=r_2^2.$$

Now
$$\begin{aligned} PQ^2&=(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2\\ &=(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)-2(x_1x_2+y_1y_2+z_1z_2)\\ &=r_1^2+r_2^2-2(l_1l_2r_1r_2+m_1m_2r_1r_2+n_1n_2r_1r_2)\\ &=r_1^2+r_2^2-2r_1r_2(l_1l_2+m_1m_2+n_1n_2). \end{aligned}$$

By trigonometry, $\quad \cos\theta=\dfrac{OP^2+OQ^2-PQ^2}{2OP.OQ}$

or $$\cos\theta=\frac{r_1^2+r_2^2-r_1^2-r_2^2+2r_1r_2(l_1l_2+m_1m_2+n_1n_2)}{2r_1r_2}$$

or $$\cos\theta=\frac{2r_1r_2(l_1l_2+m_1m_2+n_1n_2)}{2r_1r_2}.$$

Hence $$\boxed{\cos\theta=l_1l_2+m_1m_2+n_1n_2.}$$

This gives the required angle.

Corollary 1. To find the condition of perpendicularity of two lines in space.

Here $\theta=90^\circ$, then $\cos\theta=\cos 90^\circ=0$.

But $\cos\theta=l_1l_2+m_1m_2+n_1n_2$.

Hence the necessary and sufficient condition that the two lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are at right angles is

$$\boxed{l_1l_2+m_1m_2+n_1n_2=0.}$$

Corollary 2. To find $\sin\theta$, where θ is the angle between two lines with d. cs. (l_1, m_1, n_1) and (l_2, m_2, n_2).

The well known Lagrange's Identity of Algebra,

$$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$$

can be easily verified.

If (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of two lines, then $l_1^2+m_1^2+n_1^2=1$, $l_2^2+m_2^2+n_2^2=1$

and $$\cos\theta=l_1l_2+m_1m_2+n_1n_2.$$

Substituting these values in the Lagrange's Identity, we get

$$1-\cos^2\theta=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$$

or $$\sin^2\theta=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$$

or $$\boxed{\sin\theta=\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}.}$$

This gives $\sin\theta$.