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B. Sc.(honours) Part 2paper 3

Subject:Mathematics

Topic:Rolle's Theorem

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Rolle's Theorem

If a function $f(x)$ defined in a closed interval $[a, b]$ satisfies the following three conditions :

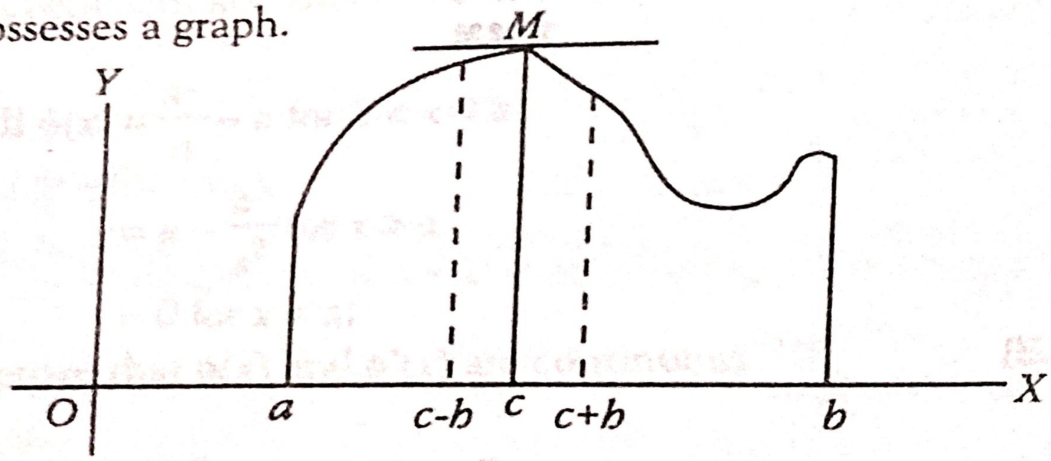
(i) $f(x)$ is continuous in the closed interval $[a, b]$.

(ii) $f'(x)$ exists (finite or infinite) for every point in the open interval $]a, b[$.

(iii) $f(a) = f(b)$,

then there exists at least one point c where $a < c < b$ i.e. $c \in]a, b[$ such that $f'(c) = 0$.

Proof : For the sake of understanding the theorem we suppose that $f(x)$ possesses a graph.



The condition (ii) of the theorem stipulates that the graph of $f(x)$ possesses a tangent at every point of the interval $]a, b[$ and the conclusion of the theorem is that under the given conditions there is a point c such that $a < c < b$ at which the tangent is parallel to the x-axis.

Since $f(x)$ is continuous in the closed interval $[a, b]$, $f(x)$ is bounded and attains its bounds at least once in $[a, b]$.

Let its least upper bound and greatest lower bound be M and m respectively.

We know that $M \geq m$, i.e. $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Case I. Suppose that $M = m$.

In this case

$$f(a) = f(\alpha) = f(\beta) = f(b) = M = m \text{ for all } \alpha, \beta \in [a, b].$$

This means that $f(x)$ is constant in $[a, b]$.

Consequently at point $c \in]a, b[$ we should have $f'(c) = 0$.

Case II. Suppose $M > m$.

In this case at least one of the bounds is different from $f(a) = f(b)$ and is attained at a point, say c other than the end points $x = a$ and $x = b$, otherwise $f(x)$ will be constant as in case I.

Here we shall prove that $f'(c) = 0$.

For the sake of definiteness, we take $f(c) = M$.

Since M is the l.u.b., therefore from the definition, it follows that $f(c+h) \leq f(c) = M$ and also $f(c-h) \leq f(c) = M$

where h is positive number such that $c \pm h \in]a, b[$.

Now, $$f(c+h) \leq f(c) \Rightarrow f(c+h) - f(c) \leq 0$$

$$\Rightarrow \frac{f(c+h) - f(c)}{h} \leq 0. \qquad ...(1)$$

and $$f(c-h) \leq f(c) \Rightarrow f(c-h) - f(c) \leq 0,$$

$$\Rightarrow \frac{f(c-h) - f(c)}{-h} \geq 0. \qquad ...(2)$$

Now taking the limits of (1) and (2) as $h \to 0$, we get

$$\underset{h \to 0}{Lt} \frac{f(c+h) - f(c)}{h} \leq 0$$

i.e. $$Rf'(c) \leq 0 \qquad ...(3)$$

and $$\underset{h \to 0}{Lt} \frac{f(c-h) - f(c)}{-h} \geq 0$$

i.e. $$Lf'(c) \geq 0. \qquad ...(4)$$

Now from the condition (ii) of the theorem we note that $f'(x)$ exists for all $x \in]a, b[$ and hence we should have $Rf'(c) = Lf'(c)$ since $c \in]a, b[$.

Hence it follows from (3) and (4) that this is possible only when

$$Rf'(c) = 0 = Lf'(c).$$

Hence $\qquad f'(c) = 0.$

Similarly when $f(c) = m$, we can prove that $f'(c) = 0$.

Q Verify Rolle's Theorem in the case of the following functions :

(i) $f(x) = \sin x$ in $[0, \pi]$

(ii) $f(x) = 3x^4 - 4x^2 + 5$ in the interval $[-1, 1]$

(iii) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

Soln. (i) The function $f(x) = \sin x$ is continuous and differentiable in the interval $[0, \pi]$.

Also, $f(0) = \sin 0 = 0$ and $f(\pi) = \sin \pi = 0$ so that $f(0) = f(\pi) = 0$.

Hence all the conditions of Rolle's theorem are satisfied.

Now, we have $f'(x) = \cos x$.

Thus $\quad f'(x) = 0 \Rightarrow \cos x = 0$

$$\Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$$

Since $x = \frac{\pi}{2}$ lies in the interval $[0, \pi]$, therefore Rolle's theorem is verified.

(ii) The function is obviously continuous and differentiable in the interval $[-1, 1]$.

Also, $f(-1) = f(1) = 4$.

Hence all the conditions of Rolle's theorem are satisfied in this interval.

Now, we have $f'(x) = 12x^3 - 8x$.

Thus $f'(x) = 0 \Rightarrow 12x^3 - 8x = 0$ i.e. $x = 0, \pm\sqrt{2/3}$.

Since all these values belong to the interval $[-1, 1]$, therefore Rolle's theorem is verified.

(iii) Here $f(x) = e^x(\sin x - \cos x)$.

The function $f(x)$ is obviously continuous and differentiable in the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Also, $$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right)$$

$$= e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

and $$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right)$$

$$= e^{\pi/4}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0.$$

$\therefore$ $$f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0.$$

Hence all the conditions of Rolle's theorem are satisfied.

Now, we have

$$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x) = 2e^x \cdot \sin x$$

Thus $f'(x) = 0 \Rightarrow e^x \sin x = 0 \Rightarrow \sin x = 0 \quad (\because e^x \neq 0)$

$$\Rightarrow x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

Out of these values $x = \pi$ lies in the given interval. Thus Rolle's theorem is verified.

Q ***Verify Rolle's theorem in the case of the following function :***

$$f(x) = 2x^3 + x^2 - 4x - 2.$$

Soln. Since $f(x) = 2x^3 + x^2 - 4x - 2$ is a polynomial in x, it is continuous and differentiable for all real values of x.

Hence first two conditions of Rolle's theorem are satisfied in any interval.

Now we need to determine two points $x = a$ and $x = b$ such that $f(a) = f(b)$.

For this, we consider the roots of the equation $f(x) = 0$, so that if α and β are the roots of $f(x) = 0$, then $f(\alpha) = f(\beta) = 0$.

Now $\quad f(x) = 0 \Rightarrow 2x^3 + x^2 - 4x - 2 = 0$

$$\Rightarrow (x^2 - 2)(2x + 1) = 0 \Rightarrow x = \pm\sqrt{2}, -\frac{1}{2}$$

Thus $\quad f(\sqrt{2}) = f(-\sqrt{2}) = f(-\frac{1}{2}) = 0.$

We take the interval $[-\sqrt{2}, \sqrt{2}]$ so that in this interval all the conditions of Rolle's theorem are satisfied.

We need to verify that $f'(x) = 0$ at least once in open interval $]-\sqrt{2}, \sqrt{2}[$.

Now $\quad f'(x) = 6x^2 + 2x - 4.$

Therefore $\quad f'(x) = 0 \Rightarrow 6x^2 + 2x - 4 = 0 \Rightarrow 3x^2 + x - 2 = 0$

$$\Rightarrow (3x - 2)(x + 1) = 0 \Rightarrow x = -1, \frac{2}{3}$$

Thus $f'(-1) = 0$ and $f'(\frac{2}{3}) = 0$.

Since both the points $x = -1$ and $x = \frac{2}{3}$ lie in the open interval $]-\sqrt{2}, \sqrt{2}[$, therefore Rolle's theorem is verifed.

Q. *Verify Rolle's Theorem in the interval $[a, b]$ for the function $f(x) = (x - a)^m(x - b)^n$, m, n being positive integers and find a suitable point c.*

Soln. The function is obviously continuous and differentiable in $[a, b]$ and $f(a) = f(b) = 0$. Thus all the conditions of Rolle's theorem are satisfied.

Hence there is a point $c \in]a, b[$ at which $f'(c) = 0$.

Now, $\quad f'(x) = m(x - a)^{m-1}(x - b)^n + (x - a)^m \cdot n(x - b)^{n-1}$

$$= (x - a)^{m-1}(x - b)^{n-1}\{m(x - b) + n(x - a)\} = 0.$$

Hence $f'(x) = 0 \Rightarrow x = a, b, \frac{mb + na}{m + n}.$

Out of these values $\frac{mb + na}{m + n} \in [a, b]$.

Hence we take $c = \frac{mb + na}{m + n}$.