B.Sc. (Math) part - III

paper VI

Topic :- Direct product of two groups

External Direct product

Def :- If G_1 and G_2 be two groups then the set of all ordered pairs $\{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ is called the external direct product of G_1 and G_2 and written as $G_1 \times G_2$

$$G_1 G_2 = \{g_1 g_2 : g_1 \in G_1, g_2 \in G_2\}$$

Theorem If G_1 and G_2 be any two abstract groups then the set $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ is a group with respect to the binary operation denoted multiplicatively and defined as

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$$

where $g_1 h_1 \in G_1$, $g_2 h_2 \in G_2$

proof :- To prove $G_1 \times G_2$ is a group we have to satisfy the following ~~operation~~ properties

(i) Closure property :— since G_1 and G_2 be the two groups

$\therefore g_1, h_1 \in G_1 \Rightarrow g_1h_1 \in G_1$

and $g_2, h_2 \in G_2 \Rightarrow g_2h_2 \in G_2$

$\therefore (g_1h_1, g_2h_2) \in G_1 \times G_2$

$\therefore G_1 \times G_2$ is closed

(ii) Associativity

$[(g_1, g_2)(h_1, h_2)](k_1, k_2)$

$= (g_1h_1, g_2h_2)(k_1, k_2)$

$= [(g_1h_1)k_1, (g_2h_2)k_2]$

$= [g_1(h_1k_1), g_2(h_2k_2)]$

$= (g_1, g_2)(h_1k_1, h_2k_2)$

$= (g_1, g_2)[(h_1, h_2)(k_1, k_2)]$

Hence the composition is associative

(iii) Existence of identity

If e_1, e_2 be the ide of groups G_1 and G_2 respec then $(e_1, e_2) \in G_1 \times G_2$

Also $g_1e_1 = e_1g_1 = g_1$

and $g_2e_2 = e_2g_2 = g_2$

Now $(g_1, g_2)(e_1, e_2) = (g_1 e_1, g_2 e_2)$
$= (g_1, g_2)$

and $(e_1, e_2)(g_1, g_2) = (e_1 g_1, e_2 g_2)$
$= (g_1, g_2)$

$\therefore (e_1, e_2) \in G_1 \times G_2$ is the identity element of $G_1 \times G_2$

(iv) **Existence of inverse**

~~let $g_1, g_2 \in G_1$~~

let $g_1 \in G_1$, $g_2 \in G_2$

$g_1 \in G_1 \Rightarrow g_1^{-1} \in G_1$

$g_2 \in G_2 \Rightarrow g_2^{-1} \in G_2$

and $(g_1^{-1}, g_2^{-1}) \in G_1 \times G_2$

Also $g_1 g_1^{-1} = g_1^{-1} g_1 = e_1$

and $g_2 g_2^{-1} = g_2^{-1} g_1 = e_2$

$(g_1, g_2)(g_1^{-1}, g_2^{-1})$
$= (g_1^{-1}, g_2^{-1})(g_1, g_2)$
$= (e_1, e_2)$ the identity by (3)

$\Rightarrow (g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}) \in G_1 \times G_2$

Thus inverse exists and belong to the set

Hence $G_1 \times G_2$ is a group with respect to binary composition.