e-content for students

B. Sc.(honours) Part 1 paper 2

Subject:Mathematics

Topic:Radius of curvature for polar equation

RRS college mokama

Radius of curvature of Polar curves r = f (θ):

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} \qquad \left(where\ r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}\right)$$

Example 9 Prove that for the cardioide r = a (1 + cos θ),

$\frac{\rho^2}{r}$ is const.

Solution: Here r = a (1+ cos θ)

$$\Rightarrow r_1 = -\text{a Sin}\,\theta \text{ and } r_2 = -\text{a cos}\,\theta$$

$$\therefore r^2 + r_1^2 = a^2\,[(1+\cos\theta)^2 + sin^2\theta] = 2a^2\,(1 + \cos\theta)$$

$$r^2 + 2r_1^2 - rr^2 = a^2[(1+\cos\theta)^2 + 2sin^2\theta + \cos\theta(1+\cos\theta)]$$

$$= 3a^2\,(1+\cos\theta)$$

$$\therefore \rho^2 = \frac{(r^2+r_1^2)^3}{(r^2+2r_1^2-rr_2)^2} = \frac{8a^6(1+cos\theta)^3}{9a^4\,(1+\cos\theta)^2} = \frac{8}{9}a^2\,(1+\cos\theta)$$

$$\Rightarrow \rho^2 = \frac{8a}{9}\text{r}$$

$\therefore \frac{\rho^2}{r} = \frac{8a}{9}$ which is a constant.

Example 10 Show that at the point of intersection of the curves r = a θ and r θ = a, the curvatures are in the ratio 3:1 ($0 < \theta < 2\pi$)

Solution: The points of intersection of curves r = a θ & r θ = a are given by a θ^2 = a or $\theta = \pm 1$
Now for the curve r=a θ we have r_1 = a and r_2 = 0

$$\therefore \text{At } \theta = \pm 1, \rho = \left[\frac{(r^2+r_1^2)^{3/2}}{2a^2+a^2\theta^2-0}\right]_{\theta=\pm 1} = \frac{a\,(2\sqrt{2})}{3} = \rho_1$$

For the curve r θ = a,

$$r_1 = \frac{-a}{\theta^2} \quad \text{and} \quad r_2 = \frac{2a}{\theta^3}$$

$$\text{At } \theta = \pm 1,\ \rho = \left[\frac{\left(\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right)^{3/2}}{\frac{2a^2}{\theta^4}+\frac{a^2}{\theta^2}-\frac{2a^2}{\theta^4}}\right]_{\theta=\pm 1,} = \left[a\frac{(1+\theta^2)^{3/2}}{\theta^4}\right]_{\theta=\pm 1}$$

$$= 2a\sqrt{2} = \rho_2$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{2a\sqrt{2}}{2a\sqrt{2/3}} = \frac{3}{1}$$

$$\therefore \rho_2 : \rho_1 = 3 : 1$$

Example 11 Find the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\theta$

Solution: $r^m = a^m \cos m\theta$

$$\Rightarrow m\log r = m\log a + \log\cos m\theta$$

$$\Rightarrow \frac{m}{r} r_1 = -m\frac{\sin m\theta}{\cos m\theta} \quad \text{(on differentiating w.r.t. } \theta)$$

$$\Rightarrow r_1 = -r\tan m\theta \quad ....(1)$$

$$\text{Now } r_2 = -(r_1 \tan m\theta + rm\sec^2 m\theta)$$

$$= r\tan^2 m\theta - rm\sec^2 m\theta \quad \text{(from (1))}$$

$$\therefore \rho = \frac{(r^2+r^2\tan^2 m\theta)^{3/2}}{r^2+2r^2\tan^2 m\theta - r^2\tan^2 m\theta + r^2 m\sec^2 m\theta}$$

$$= \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta + r^2 m\sec^2 m\theta} \qquad = \frac{r}{m+1}\sec m\theta$$

Example 12 Show that the radius of curvature at the point (r, θ)

of the curve $r^2 \cos 2\theta = a^2$ is $\frac{r^3}{a^2}$

Solution: $r^2 = a^2 \sec 2\theta$

$$\Rightarrow 2rr_1 = 2a^2 \sec 2\theta \tan 2\theta$$
$$\Rightarrow \quad r_1 = r \tan 2\theta$$

and $r_2 = 2r\, sec^2\theta + r_1 tan2\theta$

$$= 2r \sec^2 2\theta + r \tan^2 2\theta \quad (\because r = r \tan 2\theta)$$

Now $\rho = \frac{(r^1+r_1^2)^{3/2}}{2r_1^2 + r^2 - rr_2} \Rightarrow \rho = \frac{\left((r^2+r^2\tan^2 2\theta)\right)^{3/2}}{2r^2\tan^2 2\theta + r^2 - r^2(2\sec^2 2\theta + \tan^2 2\theta)}$

$$= \frac{(r^2\sec^2 2\theta)^{3/2}}{r^2(2\tan^2 2\theta + 1 - 2\sec^2 2\theta - \tan^2 2\theta)}$$

$$= \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta}$$

$$= r \sec 2\theta$$

$$= r \cdot \frac{r^2}{a^2} = \frac{r^3}{a^2}$$

Formula for polar Equation

(i) *To find the radius of curvature for the polar curve* $r=f(\theta)$.

or, *To prove the formula :* $\rho=\dfrac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}$, *where the symbols have their usual meaning.* '84)

Proof. We know that $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. .. (1)

Differentiating both sides with regard to θ, we get

$$-\frac{2}{p^3}\cdot\frac{dp}{d\theta}=-\frac{2}{r^3}\cdot\frac{dr}{d\theta}-\frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^3+\frac{2}{r^4}\cdot\frac{dr}{d\theta}\cdot\frac{d^2r}{d\theta^2}$$

or $$\frac{dp}{dr}=\left\{\frac{1}{r^3}+\frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^4}\cdot\frac{d^2r}{d\theta^2}\right\}p^3$$

$$=\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}\cdot\frac{p^3}{r^5}. \qquad \ldots (2)$$

From (1), $$\frac{1}{p^2}=\frac{r^2+\left(\frac{dr}{d\theta}\right)^2}{r^4}$$

or $$p=\frac{r^2}{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}. \qquad \ldots (3)$$

We know that $$\rho=r\frac{dr}{dp}$$

$$=r.\frac{r^5}{p^3\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}}, \text{ from (2)}$$

$$=\frac{r^6.\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^6\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}}.$$

Hence $$\boxed{\rho=\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}.}$$