e content for students of patliputra university

B. Sc. (Honrs) Part 1paper 1

Subject:Mathematics

Topic:Solution of cubic equations by Carson's method

Solution of cubic Equations

Carson's method

Let the cubic equation be $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ ...(1)

Let the given equation be transformed into the form

$$z^3 + 3Hz + G = 0 \qquad ...(2)$$

where $z = a_0x + a_1$ and H and G have their usual meanings.

Let us assume $z = p^{1/3} + q^{1/3}$ as the solution of the equation (2).

[We could have also used $z = p + q$ as the solution of (2).]

Cubing this, we get $z^3 = p + q + 3p^{1/3}\, q^{1/3}\, (p^{/3} + q^{1/3})$

$$= p + q + 3p^{1/3}q^{1/3}z$$

or, $z^3 - 3p^{1/3}\, q^{1/3}z - (p + q) = 0.$ ...(3)

Since (2) and (3) are identical, therefore comparing their coefficients, we get

$$-3p^{1/3}q^{1/3} = 3H \text{ i.e., } p^{1/3}q^{1/3} = -H \quad \therefore \quad pq = -H^3;$$

and $\quad -(p + q) = G \quad \therefore \quad p + q = -G.$

Now, $p + q = -G$ and $pq = -H^3$.

Therefore p, q are the roots of the quadratic equation

$$t^2 + Gt - H^3 = 0 \qquad ...(4)$$

Solving (4), we get $t = \dfrac{-G \pm \sqrt{G^2 + 4H^3}}{2}$.

$$\left.\begin{aligned} \text{Therefore we can take } p &= \frac{-G + \sqrt{G^2 + 4H^3}}{2} \\ \text{and} \qquad q &= \frac{-G - \sqrt{G^2 + 4H^3}}{2} \end{aligned}\right\} \qquad ...(5)$$

Lastly, by extracting the cube roots of p and q and substituting these in $z = p^{1/3} + q^{1/3}$ we can find out the roots of the equation (2) and consequently of the equation (3).

Now, the cube roots of p are $p^{1/3}, wp^{1/3}, w^2p^{1/3}$ and the cube roots of q are $q^{1/3}, wq^{1/3}, w^2q^{1/3}$; where w is the cube root of unity. Thus if we take all possible combinations of $p^{1/3}$ and $q^{1/3}$, from the sets $\{p^{1/3}, wp^{1/3}, w^2p^{1/3}\}$ and $\{q^{1/3}, wq^{1/3}, w^2q^{1/3}\}$ there shall be nine values of the expression $p^{1/3} + q^{1/3}$.

Thus it would seem that there are nine roots of the equation (2). But one should remember that these are restricted by the relation $p^{1/3}q^{1/3} = -H$.

Thus if we take $p^{1/3}$ from the first set, the corresponding element in the second set will be $q^{1/3}$ {and not $wq^{1/3}$; or $w^2q^{1/3}$; for $p^{1/3}wq^{1/3}$ or $p^{1/3}w^2q^{1/3} \neq p^{1/3}q^{1/3}$}.

Similarly, the corresponding element of $wp^{1/3}$ will be $w^2q^{1/3}$, for $wp^{1/3}\cdot w^2q^{1/3} = w^3p^{1/3}q^{1/3} = p^{1/3}q^{1/3}$ and the corresponding element of $w^2p^{1/3}$ will be $wq^{1/3}$.

Thus there shall be three and only three combinations which shall satisfy $p^{1/3}q^{1/3} = -H$; namely $(p^{1/3}, q^{1/3})$, $(wp^{1/3}, w^2q^{1/3})$ and $(w^2p^{1/3}, wq^{1/3})$.

Hence there shall be three and only three value for z, namely $z = p^{1/3} + q^{1/3},\ wp^{1/3} + w^2q^{1/3},\ w^2p^{1/3} + wq^{1/3}$.

From these values of z, the values of x can be obtained by the relation $z = a_0x + a_1$.

Thus the complete solution of the cubic is obtained.

Q. Solve $x^3 + x^2 - 16x + 20 = 0$

Soln. Comparing the given cubic with

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

we find $a_0 = 1, a_1 = 1/3, a_2 = -16/3$ and $a_3 = 20$.

$$\therefore \quad H = a_0a_2 - a_1^2 = 1\left(-\frac{16}{3}\right) - \frac{1}{9} = -\frac{49}{9}$$

and $G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$

$$= 1\cdot(20) - 3\cdot 1\left(\frac{1}{3}\right)\left(-\frac{16}{3}\right) + 2\left(\frac{1}{3}\right)^3$$

$$= 20 + \frac{16}{3} + \frac{2}{27} = \frac{540 + 144 + 2}{27} = \frac{686}{27}.$$

Hence the given cubic can be reduced to the standard form

$$z^3 + 3Hz + G = 0$$

i.e., $z^3 - 3\left(-\frac{49}{9}\right)z + \frac{686}{27} = 0$ i.e., $z^3 - \frac{49}{3}z + \frac{686}{27} = 0$

where $z = a_0x + a_1 = x + \frac{1}{3}$.

Let $z = p^{1/3} + q^{1/3}$.

Then p and q are the roots of the equation

$$t^2 + Gt - H^3 = 0$$

i.e., $$t^2 + \frac{686}{27}t - \left(-\frac{49}{9}\right)^3 = 0 \Rightarrow t^2 + \frac{686}{27}t + \frac{(49)^3}{(27)^2} = 0$$

$$\Rightarrow \quad (27)^2t^2 + (686 \times 27)t + (49)^3 = 0$$

$$\therefore \quad t = \frac{-(686 \times 27) \pm \sqrt{(686 \times 27)^2 - 4(27)^2(49)^3}}{2(27)^2}$$

$$= \frac{-686 \pm \sqrt{(686)^2 - 4(49)^3}}{2 \times 27}$$

$$= \frac{-686 \pm \sqrt{(7 \times 7 \times 7 \times 2)^2 - 4(7 \times 7)^3}}{54}$$

$$= \frac{-686 \pm \sqrt{7^6 \cdot 2^2 - 2^2 \cdot 7^6}}{54} = \frac{-686 \pm 0}{54}$$

$$= \frac{-686}{54} = -\frac{343}{27} = \left(-\frac{7}{3}\right)^3.$$

$$\therefore \quad p^{1/3} = -\frac{7}{3} \text{ and } q^{1/3} = -\frac{7}{3}.$$

Hence $z_1 = p^{1/3} + q^{1/3} = -\frac{7}{3} - \frac{7}{3} = -\frac{14}{3}$,

$$z_2 = wp^{1/3} + w^2q^{1/3} = (w + w^2)\left(-\frac{7}{3}\right) = (-1)\left(-\frac{7}{3}\right) = \frac{7}{3}$$

$$z_3 = w^2p^{1/3} + wq^{1/3} = (w^2 + w)\left(-\frac{7}{3}\right) = (-1)\left(-\frac{7}{3}\right) = \frac{7}{3}$$

Therefore from $z = x + \frac{1}{3}$ we get $x = z - \frac{1}{3}$.

Hence $x = -\frac{14}{3} - \frac{1}{3}; \frac{7}{3} - \frac{1}{3}; \frac{7}{3} - \frac{1}{3} = -5, 2, 2.$