e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Topic:Cosets of a group

Cosets of a group

Let H be a subgroup of a group G, which may be of finite or infinite order. We exhibit two partitions of G by defining two equivalence relations $\sim_L$ and $\sim_R$ on G as follows: $a \sim_L b$ if and only if $a^{-1}b \in H$ and $a \sim_R b$ if and only if $ab^{-1} \in H$. It is easy to see that these relations are reflexive, symmetric and transitive and hence they are equivalence relations on G. Now, let $a \in G$. Then the equivalence class (corresponding to $\sim_L$) containing a consists of all $x \in G$ such that $a^{-1}x \in H$. $\implies a^{-1}x = h$ for some $h \in H$. $\implies x = ah$ for some $h \in H$. Thus the equivalence class of a under the equivalence relation $\sim_L$ is the set $\{ah | h \in H\}$ which we denote by aH. Similarly, we find that the equivalence class of a under the equivalence relation $\sim_R$ is the set $\{ha | h \in H\}$ and we denote this set by Ha.

We call aH, the **left coset** of H containing a, and Ha, the **right coset** of H containing a.

For example, consider the subset $3\mathbb{Z}$ of $\mathbb{Z}$. Here we use additive notation, so the left coset containing the integer n is $n + \mathbb{Z}$. When $n = 0$, $0 + 3\mathbb{Z} = 3\mathbb{Z} = \{..., -9, -6, -3, 0, 3, 6, 9, ...\}$. Similarly, the left coset containing the integer 1 is $1 + 3\mathbb{Z} = \{..., -8, -5, -2, 1, 4, 7, 10, ...\}$ and the left coset containing 2 is $2 + 3\mathbb{Z} = \{..., -7, -4, -1, 2, 5, 8, 11, ...\}$. We note that these three left cosets constitute a partition of $\mathbb{Z}$. Since the group $\mathbb{Z}$ is abelian, the left coset $n + 3\mathbb{Z}$ is the same as the right coset $3\mathbb{Z} + n$, thus the partition of $\mathbb{Z}$ into right cosets is the same. (Generally, the left and the right coset of a subgroup determined by the same element need not be equal.)

From above example, we have the following observation.

For a subgroup H of an abelian group G, the partition of G into left cosets of H is the same as the partition of G into right cosets of H.

Let H be a subgroup of G. Now we show that every left coset and right coset of H have the same number of elements as H. We show this by exhibiting a one to one map of H onto a left coset gH of H for a fixed element g of G. Define $\phi : H \to gH$ by $\phi(h) = gh, \forall h \in H$. Since $gH = \{gh|h \in H\}$, it is clear that ϕ is onto. Now let $\phi(h_1) = \phi(h_2)$ for some $h_1, h_2 \in H$. $\implies gh_1 = gh_2$. By left cancellation, we get $h_1 = h_2$. Thus ϕ is one to one. This shows that every left coset of H have the same number of elements as H. In a similar way, we can get a one to one map of H onto the right coset Hg. Thus, ***every coset(left or right) of a subgroup H of a group G has the same number of elements as H.***

Problem

Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

Solution.

Since $2\mathbb{Z}$ is abelian, the left cosets and the right cosets are the same. The left coset containing the integer 0 is $0 + 4\mathbb{Z} = \{..., -8, -4, 0, 4, 8, ...\}$ and left coset containing the integer 2 is $2 + 4\mathbb{Z} = \{..., -6, -2, 2, 6, 10, ...\}$. Since these two left cosets exhausts $2\mathbb{Z}$, they form a partition of $2\mathbb{Z}$.

Problem

Find all cosets of the subgroup $\langle 4 \rangle$ of $\mathbb{Z}_{12}$.

Solution.

The cosets are $0 + \langle 4 \rangle = \langle 4 \rangle = \{0, 4, 8\}$, $1 + \langle 4 \rangle = \{1, 5, 9\}$, $2 + \langle 4 \rangle = \{2, 6, 10\}$, and $3 + \langle 4 \rangle = \{3, 7, 11\}$.

Problem

Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. Show that every left coset gH is the same as the right coset Hg.

Solution.

We show that $gH = Hg$ by showing that each coset is a subset of the other. Let $gh \in gH$ where $g \in G$ and $h \in H$. Then $gh = ghg^{-1}g = [(g^{-1})^{-1}hg^{-1}]g$ is in Hg because $(g^{-1})^{-1}hg^{-1}$ is in H by hypothesis. Thus gH is a subset of Hg. Now let $hg \in Hg$ where $g \in G$ and $h \in H$. Then $hg = gg^{-1}hg = g(g^{-1}hg)$ is in gH because $g^{-1}hg \in H$ by hypothesis. Thus Hg is a subset of gH also, which shows that $gH = Hg$.

Problem

Let H be a subgroup of a group G. Show that the number of left cosets of H is the same as the number of right cosets of H.

Solution. We prove this by exhibiting a one-to-one map between the collection of left cosets of H and the collection of right cosets of H. For any $a \in G$, we claim that Ha^{-1} consists of all inverses of elements in aH . For proving this, note that since H is a subgroup, we have $\{h^{-1}|h \in H\} = H$. Therefore, $Ha^{-1} = \{ha^{-1}|h \in H\} = \{h^{-1}a^{-1}|h \in H\} = \{(ah)^{-1}|h \in H\}$. This proves that Ha^{-1} consists of all inverses of elements in aH . Define a map ϕ from the collection of left cosets of H into the collection of right cosets of H by $\phi(aH) = Ha^{-1}$. Then ϕ is well defined for if $aH = bH$, then $\{(ah)^{-1}|h \in H\} = \{(bh)^{-1}|h \in H\}$. Because Ha^{-1} may be any right coset of H, the map is onto the collection of right cosets. Because elements in disjoint sets have disjoint inverses, we see that ϕ is one to one. Thus there are the same number of left as right cosets of a subgroup H of a group G.