# Planck's law and Radiation Pressure

**e-content for B.Sc Physics (Honours)**
**B.Sc Part-I**
Paper-II

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## Planck's radiation law :-

Let us consider an enclosure of volume V contains electromagnetic radiation which is in equ$^{m}$ with the walls of the enclosure at temp. T. The walls can emitts or reabsorb radiation. The electromagnetic radiation contain in this encloser at temp. T may be treated as black body radiation or called blackbody. The radiation may also be regarded as a gas of photons. The photons obey BE statistics; As the spin of photon is 1. The no. of states for photon with momentum between p and p+dp is given by

$$g(p)\,dp = \frac{4\pi V}{h^3}\, p^2 dp.$$

Each photon carries energy $E = h\nu$ and momentum $p = \frac{h\nu}{c}$, Therefore.

$$g(\nu)\,d\nu = \frac{4\pi V}{c^3}\,\nu^2 d\nu$$

As there is two independent direction of polarization, we have,

$$g(\nu)\,d\nu = 2\cdot\frac{4\pi V}{c^3}\,\nu^2 d\nu$$

$$= \frac{8\pi V}{c^3}\,\nu^2 d\nu. \quad \text{--- (1)}$$

This is the no. of state lies between freq. range $\nu$ and $\nu + d\nu$. The no. of photon in the freq. range $\nu$ and $\nu + d\nu$ is given by

$$dn = \frac{g(\nu)\,d\nu}{e^{(\alpha+\beta E)} - 1} = \frac{8\pi V}{c^3}\cdot\frac{\nu^2 d\nu}{e^{(\alpha+\beta h\nu)} - 1} \quad \text{--- (2)}$$

Now the electromagnetic radiation contain in the enclosure can be emitted or reabsorb with the walls of container. So the no. of photon is not conserve. So we must put $\alpha = 0$ ($\because \mu = 0$)

we also have $\beta = \frac{1}{KT}$.

Therefore,

$$dn = \frac{g(\nu)\,d\nu}{e^{h\nu/KT} - 1} = \frac{8\pi V}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/KT} - 1}$$

The energy density in the specified energy range is given by

$$u\,d\nu = \left(\frac{dn}{V}\right)\epsilon = \left(\frac{dn}{V}\right)h\nu$$

$$= \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/KT} - 1} \quad \text{—— (3)}$$

This is planck's radiation formula.

For $h\nu \ll KT$ Then $e^{h\nu/KT} - 1 \simeq \frac{h\nu}{KT}$.

$$\therefore u\,d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{h\nu/KT}$$

$$= \frac{8\pi KT}{c^3} \nu^2 d\nu. \quad \text{—— (4)}$$

This is Rayleigh Jeans law.

When $h\nu \gg KT$, Then $e^{h\nu/KT} - 1 \simeq e^{h\nu/KT}$

$$\therefore u\,d\nu = \frac{8\pi h}{c^3} \nu^3 e^{-h\nu/KT} d\nu. \quad \text{—— (5)}$$

This Wien's law.

The total energy density is

$$\frac{U}{V} = \int_0^\infty u(\nu, T)\,d\nu.$$

$$= \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 \cancel{e^{h\nu/KT}}}{e^{h\nu/KT} - 1} d\nu$$

we put $h\nu/KT = x$

$$\therefore \frac{u}{v} = \frac{4\pi h}{c^3}\left(\frac{KT}{h}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$= \frac{8\pi K^4 T^4}{c^3 h^3} \cdot \frac{\pi^4}{15}$$

$$= \frac{8\pi^5 K^4}{15 c^3 h^3} \cdot T^4$$

$$= bT^4 \quad \left[\text{where } b = \frac{8\pi^5 K^4}{15 c^3 h^3}\right].$$

This is stefan-Boltzmann ~~...~~ Law.

Wien's displacement law :→.

Planck radiation formula is given by

$$u\, d\nu = \frac{8\pi h \nu^3}{c^3} \cdot \frac{d\nu}{e^{\alpha + \beta h\nu} - 1} \quad \text{--- (1)}$$

The above formula can be written as putting $\nu = \frac{c}{\lambda}$.

$$u(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda KT} - 1}.$$

we want to find the max$^m$ value of $\lambda$ say $\lambda = \lambda_m$ for which $u(\lambda)$ is max$^m$. For this purpose we differentiate $u(\lambda)$ with respect to $\lambda$ and set, $\left.\frac{du(\lambda)}{d\lambda}\right|_{\lambda=\lambda_m} = 0$

$$\therefore -\frac{8\pi hc \cdot 5}{\lambda_m^6} \cdot \frac{1}{e^{hc/\lambda_m KT} - 1} + \frac{\frac{8\pi hc}{\lambda_m^5} \cdot e^{hc/\lambda_m KT} \cdot \frac{hc}{\lambda_m^2 KT}}{\left(e^{hc/\lambda_m KT} - 1\right)^2} = 0.$$

$$\text{or}\quad \frac{\frac{hc}{\lambda_m KT} e^{hc/\lambda_m KT}}{e^{hc/\lambda_m KT} - 1} = 5$$

we put, $\frac{hc}{\lambda_m KT} = x$

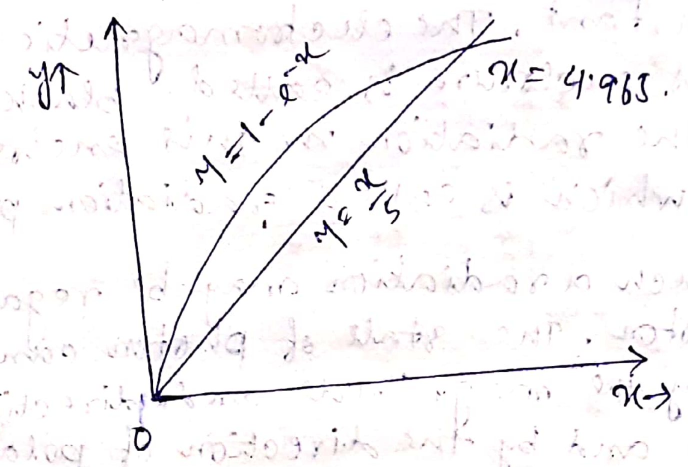
$$\therefore \frac{xe^x}{e^x - 1} = 5$$

$$\text{or, } \frac{x}{5} = 1 - e^{-x}$$

This transendental eqn. can be solve numerically and graphically. For graphically solution we put,

$$y = \frac{x}{5}$$

$$y = 1 - e^{-x}$$




From this graph we get, $x = 4.965$

$$\therefore \frac{hc}{\lambda_m KT} = 4.965$$

$$\therefore \lambda_m T = \frac{hc}{4.965K} =$$

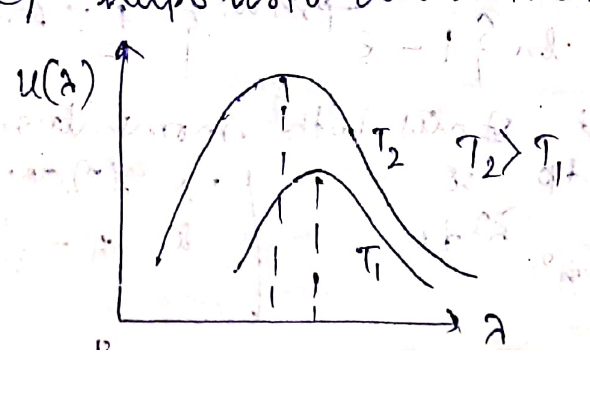
$$= \text{constant.}$$

This is Wien's displacement law.

This law states that as a temp. of black body radiation increases the max$^m$ of $u(\lambda)$ shifts or moves towards the shorter wave length in such a way that

$$\lambda_m T = \text{constant.}$$

This eqn. helps us to determine the temp of a celestial object.



## Radiation pressure :-

Let us consider an electromagnetic radiation is enclose in an enclosure of volume V of the enclosure is maintain at a temp. T. It will emitt or re-absorb photon. After a certain lapse of time, a situation will be established in which the photons and the matter of which the cavity is composed, will be in thermodynamic equm. The number of photons in such an enclosure is not definite, but its total energy remains constant. The electromagnetic radiation with in this enclosure is called black body radiation. The radiation in this enclosure exerts a pressure which is called radiation pressure.

Such a radiation may be regarded as a gas of photon. The state of photon can be specified by the magnitude and direction of its momentum and by the direction of polarisation of the electromagnetic wave associated with it. If the wave is regarded as quantised, then the photon is considered as a relativistic particle of energy $\epsilon = \hbar\omega$ and momentum $p = \frac{\hbar\omega}{c}$. The photons have integral spin and hence the photon gas obeys Bose statistics. The partition function for the assembly of photon can be written as,

$$Z = \sum_{n_i=0}^{\infty} e^{-\beta\epsilon_i n_i}$$

$$\text{or, } Z = 1 + e^{-\beta\epsilon_i} + e^{-2\beta\epsilon_i} + \cdots$$

$$\text{or, } Z = \prod_i \{1 - e^{-\beta\epsilon_i}\}^{-1}$$

$$\text{or, } \ln Z = -\sum_i \ln\{1 - e^{-\beta\hbar\omega_i}\} \quad [\because \epsilon = \hbar\omega]$$

The no. allowed states with momentum between $p$ and $p+dp$ is

$$= \frac{V}{2\pi^2\hbar^3} p^2 dp$$

$\because g(p)\,dp = \frac{4\pi V}{h^3} p^2 dp$

& $\hbar = \frac{h}{2\pi}$.

$$= \frac{V}{2\pi^2 c^3}\omega^2 d\omega \qquad \left[\because p = \frac{\hbar\omega}{c}\right]$$

However in the photon gas there is two independent direction of polarization of the electromagnetic wave perpendicular to the direction of propagation.

Therefore, the no. of allowed state

$$= \frac{V\omega^2}{\pi^2 c^3} d\omega.$$

Replacing the summation by integration, we get,

$$\ln Z = -\frac{V}{\pi^2 c^3}\int_0^\infty \omega^2 \ln\{1 - e^{-\beta\hbar\omega}\} d\omega$$

$$= \left[-\frac{V}{\pi^2 c^3}\cdot\frac{\omega^3}{3}\ln\{1-e^{-\beta\hbar\omega}\}\right]_0^\infty + \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\omega^3}{3}\frac{\beta\hbar e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} d\omega$$

The 1st term vanishes at both the limits.

Therefore,

$$\ln Z = \frac{V\beta\hbar}{3\pi^2 c^3}\int_0^\infty \frac{\omega^3}{e^{\beta\hbar\omega}-1} d\omega$$

We put, $\beta\hbar\omega = x$.

$$\therefore \ln Z = \frac{V}{3\pi^2 c^3 \beta^3\hbar^3}\int_0^\infty \frac{x^3}{e^x - 1} dx.$$

$$= \frac{V}{3\pi^2 c^3\beta^3\hbar^3}\cdot\frac{\pi^4}{15}.$$

$$= \frac{V\pi^2}{45}\left(\frac{KT}{\hbar c}\right)^3$$

The pressure is defined as

$$p = \frac{1}{\beta}\frac{\partial(\ln Z)}{\partial V} = \frac{\pi^2 KT (KT)^3}{45 (\hbar c)^3}$$

$$= \frac{\pi^2}{45}\cdot\frac{(KT)^4}{\hbar^3 c^3}.$$

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The ~~total~~ energy,

$$\cancel{E} = -\frac{\partial \ln Z}{\partial \beta}$$

$$= KT^2 \frac{\partial \ln Z}{\partial T}$$

$$= \frac{V\pi^2}{15} \cdot \frac{(KT)^4}{(\hbar c)^3}$$

Therefore energy density

$$u = \frac{E}{V} = \frac{\pi^2}{15} \frac{(KT)^4}{(\hbar c)^3}$$

$$\therefore \; u = 3 \cdot \frac{\pi^2}{45} \frac{(KT)^4}{(\hbar c)^3}$$

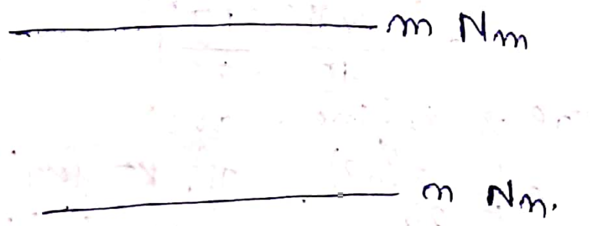
$$\therefore \; u = 3p$$

$$\therefore \; p = \frac{1}{3} u.$$

Thus the pressure is equal to $\frac{1}{3}$ of the energy density.

## Einstein's derivation of Planck's law :-

An atom emitts radiation if an electron makes a transition from a higher energy state $m$ to a lower energy state $n$ $(m \to n)$ The transition can be either spontaneous, or stimulated by the presence of an electomag-netic radiation. If $N_m$ is the no. of atoms in the state $m$. The no. of spontaneous radiation transmition per sec. is $N_m A_{mn}$ where $A_{mn}$ is the




co-efficient of ~~por~~ proportionality. On the other hand the no. of induced emission is $N_m B_{mn} u$.

where u is the external radiation density present and $B_{mn}$ is the co-efficient of proportionality for induced transition. The $A_{mn}$ and $B_{mn}$ are called Einstein A, B co-efficient. We can also have transition from m to n by external radiation. The corresponding no. of absorption per sec. will be $N_n B_{nm} u$. The spontaneous is incoherent while the stimulating radiation are in the same phase at the ext. radiation.

When thermodynamic eqn is obtain, we have,

$$N_m A_{mn} + N_m B_{mn} u = N_n B_{nm} u \quad \text{——} (1)$$

Now using Boltzmann distribution for the energy distribution of the atom we have

$$\frac{N_m}{N_n} = e^{-h\nu/KT} \quad \text{——} (2)$$

where $h\nu$ is energy difference between level m and n.

Substituting (2) in eqn (1) we get,

$$u = \frac{A_{mn}}{e^{h\nu/KT} B_{nm} - B_{mn}}$$

We put $B_{nm} = B_{mn}$ because the transition probability between the level m and n, is the matrix element of a Hermitian Hamiltonian and so symmetric.

∴ we have,

$$u = \frac{A_{mn}/B_{mn}}{e^{h\nu/KT} - 1} \quad \text{——} (3)$$

when T is very large

$$\frac{h\nu}{KT} << 1 \quad \text{ie} \quad e^{h\nu/KT} - 1 \simeq \frac{h\nu}{KT}$$

$$\therefore \quad u = \frac{A_{mn}}{B_{mn}} \left(\frac{KT}{h\nu}\right) \quad \text{——} (4)$$

Einstein compair this result with Rayleigh Jeans law $u = \frac{8\pi KT}{c^3}\nu^2$ and obtain

$$\frac{A_{mn}}{B_{mn}} = \frac{8\pi h\nu^3}{c^3} \quad \text{--- (5)}$$

Therefore, eqn (3) becomes,

$$u\,d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{e^{h\nu/KT}-1} \quad \text{--- (6)}$$

This is planck's radiation law.

The radiative transition from state m to n are given by $(A_{mn} + B_{mn}u)N_m$.

and the inverse transition $B_{nm} u N_n$. For atoms kept in a radiation field of density $u$, this incident density will emerge from the atom with intensity given by $(N_m - N_n)uB_{mn}$. It is either amplified or reduced depending upon.

$N_m > N_n$ (amplified)

$N_m < N_n$ (reduced).

To obtain the amplified radiation we must have $N_m > N_n$. This is known as population inversion. This is the principal of Laser (light amplified by stimulating emission of radiation) light. The population is done by optical pumping using cavity rasonators. For achiving the condn $N_m > N_n$. we have $\frac{N_m}{N_n} = e^{-h\nu/KT} > 1$. This can be achive when T is negative. This is the concept of negative temp.