e-content for students

B. Sc.(honours) Part 2 paper 3

Subject:Mathematics

Topic:Solved problems on continuity & differetiability

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differentiability

Ex. Prove that the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Soln. For continuity of the function at $x = 0$, we have $f(0) = 0$

$$f(0+0) = \underset{h \to 0}{\text{Lt}} f(0+h) = \underset{h \to 0}{\text{Lt}} |0+h| = 0$$

$$f(0-0) = \underset{h \to 0}{\text{Lt}} f(0-h) = \underset{h \to 0}{\text{Lt}} |0-h| = 0$$

$$\therefore \quad f(0+0) = f(0-0) = f(0).$$

Therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \underset{h \to 0}{\text{Lt}} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{|h| - 0}{h} = 1;$$

... $x = 0$: ... ing function

$$f(x) = 2 + x \text{ if } x \geq 0$$
$$= 2 - x \text{ if } x < 0.$$

Soln. For continuity, we have

$$f(0) = 2$$

$$f(0+0) = \underset{h \to 0}{Lt} f(0+h) = \underset{h \to 0}{Lt} f(h)$$

$$= \underset{h \to 0}{Lt} (2+h) = 2;$$

$$f(0-0) = \underset{h \to 0}{Lt} f(0-h) = \underset{h \to 0}{Lt} f(-h)$$

$$= \underset{h \to 0}{Lt} (2-h) = 2.$$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0}{Lt} \frac{f(h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(2+h) - 2}{h} = 1;$$

$$Lf'(0) = \underset{h \to 0}{Lt} \frac{f(0-h) - f(0)}{-h} = \underset{h \to 0}{Lt} \frac{f(-h) - f(0)}{-h}$$

$$= \underset{h \to 0}{Lt} \frac{(2+h) - 2}{-h} = -1.$$

Since $Rf'(0) \neq Lf'(0)$, therefore the function is not differentiable at $x = 0$.

Examine the continuity and differentiability of the function

$$f(x) = x \sin\frac{1}{x};\ x \neq 0$$

$$f(0) = 0$$

at the point $x = 0$.

Soln. We first of all examine the continuity of the function at $x = 0$.

We have,

$$f(0) = 0$$

$$f(0+0) = \underset{h \to 0}{Lt} f(0+h)$$

$$= \underset{h \to 0}{Lt} f(h) = \underset{h \to 0}{Lt} h \sin\frac{1}{h}$$

$$= 0, \text{ since } \left|\sin\frac{1}{h}\right| \leq 1;$$

and
$$f(0-0) = \underset{h \to 0}{Lt} f(0-h) = \underset{h \to 0}{Lt} f(-h)$$

$$= \underset{h \to 0}{Lt} h \sin\frac{1}{h} = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{h \sin\frac{1}{h}}{h} = \underset{h \to 0}{Lt} \sin\frac{1}{h}$$

which does not exist

Similarly $Lf'(0)$ does not exist.

Hence $f(x)$ is not differentiable at $x = 0$.

Ex. If $f(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$

$f(0) = 0;$

show that $f(x)$ is continuous and differentiable at $x = 0$.

Soln. For continuity of the function at $x = 0$, we have

$$f(0) = 0$$

$$f(0+0) = \underset{h \to 0}{Lt} f(0+h)$$

$$= \underset{h \to 0}{Lt} h^2 \sin\frac{1}{h} = 0; \text{ since } \left|\sin\frac{1}{h}\right| \leq 1$$

and $$f(0-0) = \underset{h \to 0}{Lt} f(0-h) = \underset{h \to 0}{Lt} h^2 \sin\frac{1}{-h} = 0.$$

Since $$f(0+0) = f(0-0) = f(0),$$

therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h \to 0}{Lt} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \underset{h \to 0}{Lt} h \sin\frac{1}{h} = 0;$$

and $$Lf'(0) = \underset{h \to 0}{Lt} \frac{f(0-h) - f(0)}{-h} = \underset{h \to 0}{Lt} \frac{h^2 \sin\frac{1}{-h} - 0}{-h}$$

$$= \underset{h \to 0}{Lt} h \sin\frac{1}{h} = 0.$$

Hence $Rf'(0) = Lf'(0)$ and consequently $f(x)$ is differentiable at $x = 0$.

Ex. 3. Examine the continuity and differentiability of the function $f(x)$ if

$$f(x) = x \cdot \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} \text{ when } x \neq 0$$

$$f(0) = 0.$$

Soln. Let $x \neq 0$.

If $x \neq 0$, then $x, e^{1/x}$ and $e^{-1/x}$ are continuous and differentiable functions and therefore $f(x)$ is continuous and differentiable when $x \neq 0$.

Now we test the continuity and differentiability of the function at $x = 0$.

We have, $|f(h) - f(0)| = \left|h\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0\right|$. ...(1)

As $e^{1/h}$ and $e^{-1/h}$ are both positive, hence

$$|e^{1/h} - e^{-1/h}| < e^{1/h} + e^{-1/h}.$$

Hence from (1), $|f(h) - f(0)| \leq |h|$.

Hence given any $\varepsilon > 0$, there exists a δ satisfying $0 < \delta < \varepsilon$ such that $|f(h) - f(0)| < \varepsilon$ provided $|h - 0| < \delta$.

Hence $f(x)$ is continuous at $x = 0$.

Again, for differentiability at $x = 0$, we have

$$Rf'(0) = \underset{h \to 0}{Lt} \frac{f(h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{1}{h}\left[h\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}} - 0\right]$$

$$= \underset{h \to 0}{Lt} \frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \underset{h \to 0}{Lt} \frac{1 - e^{-2/h}}{1 + e^{-2/h}} = \frac{1 - 0}{1 + 0} = 1.$$

$$Lf'(0) = \underset{h \to 0}{Lt} \frac{f(0 - h) - f(0)}{-h} = \underset{h \to 0}{Lt} \frac{f(-h)}{-h}$$

$$= \underset{h \to 0}{Lt} \frac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}$$

$$= \underset{h \to 0}{Lt} \frac{e^{-2h} - 1}{e^{-2/h} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Thus $Rf'(0) = 1$ and $Lf'(0) = -1$.

Hence $f'(0)$ does not exist.

Hence $f(x)$ is not differentiable at $x = 0$.

$$f(x) = x^2 \sin\frac{1}{x}; \quad x \neq 0$$
$$= 0; \quad x = 0.$$

is differentiable for every value of x but the derivative is not continuous at $x = 0$.

Soln. Let $x = c \neq 0$. Then in order to determine whether $f(x)$ is differentiable at $x = c$, we shall need to find

$$\underset{h \to 0}{Lt} \frac{f(c+h) - f(c)}{h}.$$

Now $f(c+h) = (c+h)^2 \sin\frac{1}{c+h}$ so that

$$\frac{f(c+h) - f(c)}{h} = \frac{(c+h)^2 \sin\frac{1}{c+h} - c^2 \sin\frac{1}{c}}{h}$$

$$= \frac{(c^2 + 2ch + h^2)}{h} \sin\frac{1}{c+h} - \frac{c^2}{h}\sin\frac{1}{c}$$

$$= \frac{c^2}{h}\left\{\sin\frac{1}{c+h} - \sin\frac{1}{c}\right\} + (2c+h)\sin\frac{1}{c+h}$$

$$= -\frac{c^2}{h} 2\cos\frac{2c+h}{2c(c+h)} \cdot \sin\frac{h}{2c(c+h)} + (2c+h)\sin\frac{1}{c+h}$$

$$= -2c^2 \cos\frac{2c+h}{2c(c+h)} \times \frac{\sin\frac{h}{2c(c+h)}}{\frac{h}{2c(c+h)}} \times \frac{1}{2c(c+h)}$$

$$+ (2c+h)\sin\frac{1}{(c+h)}.$$

$$\therefore \quad \underset{h \to 0}{Lt} \frac{f(c+h) - f(c)}{h} = -2c^2 \times \cos\frac{2c}{2c^2} \times 1 \times \frac{1}{2c^2} + 2c\sin\frac{1}{c}$$

$$= 2c\sin\frac{1}{c} - \cos\frac{1}{c}.$$

$$\therefore \quad f'(c) = 2c\sin\frac{1}{c} - \cos\frac{1}{c} \text{ when } c \neq 0.$$

But $$\left|h^2\sin\frac{1}{h}+h\cos\frac{1}{h}\right| \le \left|h^2\sin\frac{1}{h}\right| + \left|h\cos\frac{1}{h}\right|$$

$$\le |h^2| + |h| \quad \because \left|\sin\frac{1}{x}\right|, \left|\cos\frac{1}{x}\right| \le 1$$

$$\therefore \quad \underset{h\to 0}{Lt}\left|h^2\sin\frac{1}{h}+h\cos\frac{1}{h}\right| = 0.$$

Thus $$f(0+0) = 0.$$

$$f(0-0) = \underset{h\to 0}{Lt} f(0-h) = \underset{h\to 0}{Lt} f(-h)$$

$$= \underset{h\to 0}{Lt}\left\{-h^2\sin\frac{1}{h} - h\cos\frac{1}{h}\right\}$$

$$= -\underset{h\to 0}{Lt}\left\{h^2\sin\frac{1}{h} + h\cos\frac{1}{h}\right\}$$

$$= 0 \text{ as above.}$$

Since $f(0+0) = f(0-0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

As regards differentiability, we have

$$Rf'(0) = \underset{h\to 0}{Lt}\frac{f(0+h)-f(0)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{f(h)-f(0)}{h} = \underset{h\to 0}{Lt}\frac{f(h)}{h}$$

$$= \underset{h\to 0}{Lt}\frac{h^2\sin\frac{1}{h}+h\cos\frac{1}{h}}{h}$$

$$= \underset{h\to 0}{Lt}\left(h\sin\frac{1}{h}+\cos\frac{1}{h}\right).$$

But as we have shown before that $\underset{h\to 0}{Lt}\cos\frac{1}{h}$ does not exist.

Hence $Rf'(0)$ does not exist.

Similarly it can be shown that $Lf'(0)$ does not exist.

Hence the function is not differentiable at $x = 0$.