e-content for students

B. Sc.(honours) Part 1 paper 2

Topic :Area of curves

Subject mathematics

RRS college mokama

Area of Curves

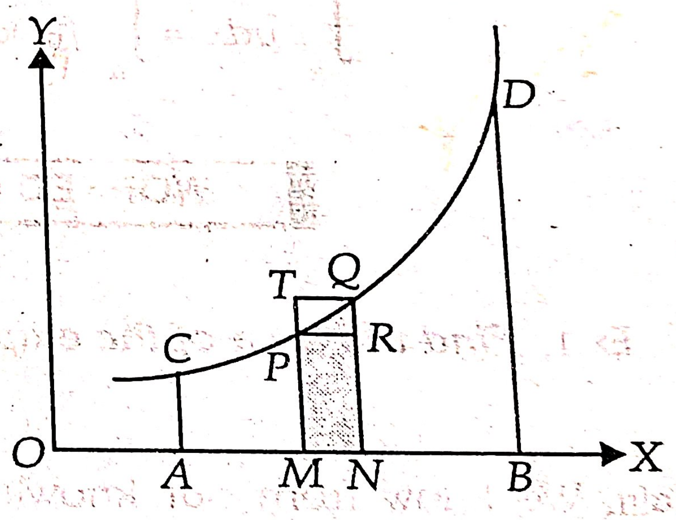
area in Cartesian co-ordinates

If $f(x)$ is a single-valued and continuous function of x in the interval $[a, b]$, then the area bounded by the curve $y = f(x)$, the x-axis and the two ordinates $x = a$ and $x = b$ is represented by

$$\int_a^b y\,dx \text{ or } \int_a^b f(x)dx.$$

Let CD be the curve given by the equation $y = f(x)$ and let AC and BD be the two ordinates at $x = a$ and $x = b$ respectively; $b > a$.

We are required to find the area bounded by the curve $y = f(x)$, the x-axis and the two ordinates at $x = a$ and $x = b$ i.e. to find the area $ABDC$.



Let P be any point on the curve whose co-ordinates are (x, y) and let $Q(x + \delta x, y + \delta y)$ be a point very close to it so that,

$$OM = x,\; PM = y,\; ON = x + \delta x,\; QN = y + \delta y$$

$$\therefore \quad MN = ON - OM = x + \delta x - x = \delta x.$$

Ex *Trace the curve* $y^2 = x(x-1)^2$ *and find the area.*

Soln. We note the following facts regarding the graph of the curve.

(i) The curve passes through the origin, since by putting $x = 0, y = 0$, both the sides of the equation are satisfied.

(ii) The curve is symmetrical about $y = 0$ (i.e. x-axis) since it contains y^2. (only even powers of y).

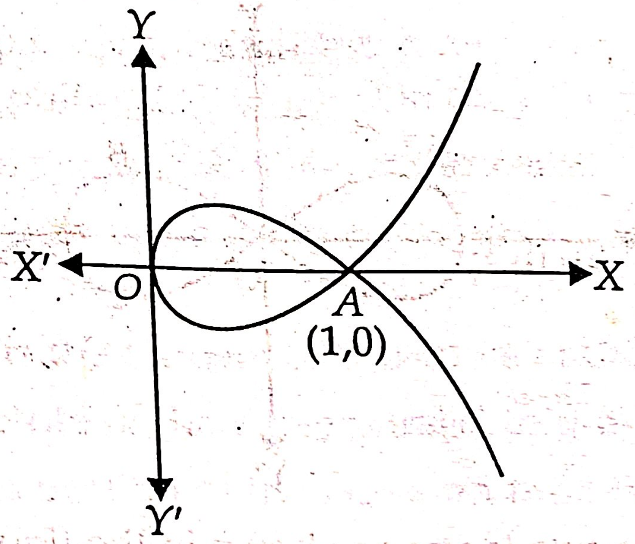
(iii) It cuts the x-axis at the points $x = 0$ and $x = 1$. This is obtained by putting $y = 0$ in the equation of the curve. It cuts the y-axis only at the origin.

(iv) If $x < 0$ (i.e. x is negative), then $y^2 = -ve$ which $\Rightarrow$ y is imaginary. That is, there is no portion of the curve on the L.H.S. of $x = 0$ (i.e. on the L.H.S. of y-axis).

Also, the value of y increases as x increases.

(v) By equating to zero the term of the lowest degree, we find that $x = 0$ which means that the y-axis is tangent to the curve at the origin.

Hence the graph of the curve will be as follows :



Thus there is a loop in the interval [0, 1].

Therefore the area of the loop

$$= 2\int_0^1 y\,dx$$

$$= 2\int_0^1 \sqrt{x}(1-x)dx;\ (\text{since } x < 1)$$

$$= 2\int_0^1 (\sqrt{x} - x^{3/2})dx$$

$$= 2\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= 2\left[\frac{2}{3} - \frac{2}{5}\right] = 2 \times \frac{4}{15} = \frac{8}{15}.$$

ex *Find the whole area of the curve* $a^2y^2 = x^2(a^2 - x^2)$.

Soln. We note the following facts regarding the graph of the given curve :

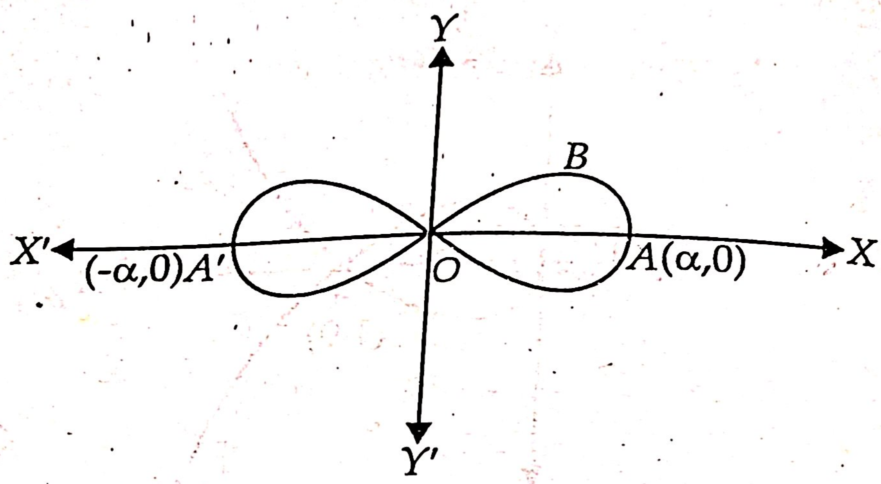
(i) The curve passes through the origin.

(ii) The curve is symmetrical about both the axes.

(iii) It cuts the x-axis at the points $x = 0$, $x = +a$, $x = -a$. It cuts the x-axis only at the point $y = 0$.

(iv) There is no portion of the curve beyond $x = +a$ or $x = -a$ for in that case y^2 is $-ve$ and hence y is imaginary.

Thus the graph of the curve will be like this :



Therefore there are two loops as shown in the figure.

Hence the area of the whole curve

$$= 4 \times \text{area } OAB$$

$$= 4\int_0^a y\,dx = 4\int_0^a \frac{x}{a}\sqrt{a^2 - x^2}\,dx \qquad \text{...(1)}$$

Put $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$.

Also $\quad x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

and $\quad x = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$.

Therefore (1) becomes

$$= 4\int_0^{\pi/2} \sin\theta \cdot a\cos\theta \cdot a\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta \qquad \text{...(2)}$$

Now let $\cos\theta = u$ so that $-\sin\theta\,d\theta = du$.

Therefore (2),

$$= 4a^2\int_1^0 u^2(-du) = 4a^2\int_0^1 u^2\,du$$

$$= 4a^2\left[\frac{u^3}{3}\right]_0^1 = \frac{4a^2}{3}.$$

Let the area $AMPC$ be denoted by A.

Then the area $ANQC$ will be denoted by $A + \delta A$.

Therefore the shaded area $PMNQ$

$$= \text{area } ANQC - \text{area } AMPC = A + \delta A - A = \delta A.$$

Now area $PMNR = y\delta x$ and area $TMNQ = (y + \delta y)dx$.

Since the area $PMNR <$ area $PMNQ <$ area $TMNQ$

therefore $y\delta x < \delta A < (y + \delta y)\delta x$.

This $\Rightarrow$ $$y < \frac{\delta A}{\delta x} < y + \delta y$$

$$\Rightarrow \quad \underset{\delta x \to 0}{Lt} \frac{\delta A}{\delta x} = y$$

$$\Rightarrow \quad \frac{dA}{dx} = y = f(x)$$

$$\Rightarrow \quad A = \int f(x)dx = F(x) + k$$

where k is a constant and $F(x)$ is an indefinite integral of $f(x)$.

Now when $x = a$, $A = 0$.

Also when $x = b$, the area A is the required area say A'.

Therefore $0 = F(a) + k$ and $A' = F(b) + k$

Hence $$A' = F(b) - F(a) = \int_a^b f(x)dx.$$

Thus the required area between the curve $y = f(x)$, the axis of x ordinates at $x = a$ and $x = b$ is

$$\int_a^b ydx = \int_a^b f(x)dx.$$