e-content for students

B. Sc.(honours) Part 2paper 3

Subject:Mathematics

Topic:Lagrange's Theorem, Taylor's Theorem

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Lagrange's Mean ValueTheorem

Statement. *If the function $f(x)$ be so defined that*

(i) *$f(x)$ is continuous at every point of the closed interval*

$$a \leqslant x \leqslant b,$$

(ii) *$f'(x)$ exists at every point of the open interval $a<x<b$, then there is a value c of x for which*

$$\frac{f(b)-f(a)}{b-a}=f'(c),$$

where $a<c<b$.

Proof. Let us consider the function $\varphi(x)$, where

$$\varphi(x)=f(b)-f(x)-\frac{b-x}{b-a}\{f(b)-f(a)\}. \qquad \ldots (1)$$

It is given that $f(x)$ is continuous in $a \leqslant x \leqslant b$ and $(b-x)$ is also continuous. We know that the algebraic sum of continuous functions is also continuous.

Hence $\varphi(x)$ must be continuous in $a \leqslant x \leqslant b$.

Differentiating (1) with respect to x, we get

$$\varphi'(x)=-f'(x)+\frac{f(b)-f(a)}{b-a}. \qquad \ldots (2)$$

It is given that $f'(x)$ exists in $a<x<b$. Therefore, $\varphi'(x)$ must exist in $a<x<b$.

Putting $x=a$ in (1), we get

$$\varphi(a)=f(b)-f(a)-\frac{b-a}{b-a}\{f(b)-f(a)\}$$
$$=f(b)-f(a)-\{f(b)-f(a)\}=0.$$

Again, putting $x=b$ in (1), we get

$$\varphi(b)=f(b)-f(b)-\frac{b-b}{b-a}\{f(b)-f(a)\}=0.$$

$\therefore \quad \varphi(a)=\varphi(b)$.

Hence the function $\varphi(x)$ under consideration satisfies all the conditions of Rolle's theorem,

viz., $\varphi(x)$ is continuous in $[a, b]$,

$\varphi'(x)$ exists in $]a, b[$,

$\varphi(a)=\varphi(b)$.

Hence, by Rolle's theorem, there exists a value $x=c$, where $a<c<b$, at which $\varphi'(c)=0$. $\qquad \ldots (3)$

Putting $x=c$ in (2), we get

$$\varphi'(c)=-f'(c)+\frac{f(b)-f(a)}{b-a}$$

$$0 = -f'(c) + \frac{f(b) - f(a)}{b - a}, \text{ from } (3)$$

or

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

or

Taylor's Theorem

Statement. *If a function $f(x)$ is defined in such a way that*

(i) *the $(n-1)$th derivative $f^{n-1}(x)$ exists and is continuous at every point of the closed interval $a \leq x \leq a+h$,*

(ii) *the nth derivative $f^n(x)$ exists at every point of the open interval $a < x < a+h$, and*

(iii) *p is any given positive integer,*

then there exists at least one number, θ, lying between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{(n-1)!\,p}f^n(a+\theta h).$$

Proof. Let us consider a function $\varphi(x)$ such that

$$\varphi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{2!}f''(x)+\ldots$$

$$+\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+A(a+h-x)^p, \quad \ldots (1)$$

where the constant A be such that

$$\varphi(a)=\varphi(a+h).$$

Putting $x=a$ in (1), we get $\quad \ldots (2)$

$$\varphi(a)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p.$$

Again, putting $x=a+h$ in (1), we get $\varphi(a+h)=f(a+h)$.

Substituting these two values in (2), we get

$$f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+Ah^p=f(a+h).$$

It is given that $f^{n-1}(x)$ is continuous in $a\leqslant x\leqslant a+h$. $\quad \ldots (3)$

Therefore, $f(x)$, $f'(x)$, $f''(x),\ldots f^{n-2}(x)$ must be continuous in $a\leqslant x\leqslant a+h$.

Hence, from (1), $\varphi(x)$ is continuous in $a\leqslant x\leqslant a+h$.

Again $\varphi'(x)$ exists in $a<x<a+h$.

Also $\quad \varphi(a)=\varphi(a+h)$.

We find that $\varphi(x)$ satisfies all the three conditions of Rolle's theorem. Hence, in accordance with Rolle's theorem, there must exist at least one number, θ, where $\theta\in\,]0, 1[$, such that

$$\varphi'(a+\theta h)=0. \quad \ldots (4)$$

Differentiating (1) with respect to x, we get

$$\varphi'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-(a+h-x)f''(x)$$

$$+\frac{(a+h-x)^2}{2!}f'''(x)+\ldots$$

$$-\frac{(a+h-x)^{n-2}}{(n-2)!}f^{n-1}(x)+\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)$$

$$=\frac{(a+h-x)^{n-1}}{(n-1)!}f^n(x)-Ap(a+h-x)^{p-1}.$$

Putting $x=a+\theta h$, we get

$$\varphi'(a+\theta h)=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}f^n(a+\theta h)-Ah^{p-1}p(1-\theta)^{p-1}$$

or $$0=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}f^n(a+\theta h)-Ah^{p-1}p(1-\theta)^{p-1}, \quad \text{from } (4)$$

or $$Ah^{p-1}p(1-\theta)^{p-1}=\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!}f^n(a+\theta h)$$

or $$A=\frac{h^{n-p}(1-\theta)^{n-p}}{(n-1)\,!\,p}f^n(a+\theta h).$$

Substituting this value in (3), we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2\,!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+\frac{h^n(1-\theta)^{n-p}}{(n-1)\,!\,p}f^n(a+\theta h).$$

Hence the theorem is proved.