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B. Sc. (Honrs) Part 2 paper 3

Subject:Mathematics

Title/Heading:Groups:L'Hospital's Rule

By Dr. Hari kant singh

Associate professor in mathematics

Rrs college mokama patna

cyclic group

Definition

Let G be a group and let $a \in G$. Then the subgroup $H = \{a^n \mid n \in \mathbb{Z}\}$ is called the **cyclic subgroup of G generated by** a, and is denoted by $\langle a \rangle$.

If the subgroup $\langle a \rangle$ of G is finite, then **order of** a is the order of $|\langle a \rangle|$ of this subgroup. Otherwise, we say that a is of **infinite order**.

Remark.

If a is of finite order m, then m is the smallest positive integer such that $a^m = e$.

Definition 1

An element a of a group G **generates** G and is a **generator for** G if $\langle a \rangle = G$.

A group G is **cyclic** if there is some element $a \in G$ that generates G.

Example

1. The group $\mathbb{Z}_4$ is and $\langle 1 \rangle = \langle 3 \rangle = \mathbb{Z}_4$. Also, $\langle 2 \rangle = \{0, 2\}$ and $\langle 0 \rangle = \{0\}$.
2. The Klein -4 group V is not cyclic, since $\langle e \rangle = \{e\}, \langle a \rangle = \{e, a\}, \langle b \rangle = \{e, b\}$ and $\langle c \rangle = \{e, c\}$.
3. The group $\langle \mathbb{Z}, + \rangle$ is cyclic. Both 1 and -1 are generators for this group and they are the only generators. (Why?)
4. Consider the group $\langle \mathbb{Z}, + \rangle$. Note that the cyclic subgroup generated by $n \in \mathbb{Z}$ consists of all multiples of n. i.e., $\langle n \rangle = n\mathbb{Z}$.

Problem

Show that every cyclic group is abelian.

Solution.

Let G be cyclic and let a be a generator for G. For $x, y \in G$, there ex $m, n \in \mathbb{Z}$ such that $x = a^m$ and $y = a^n$. Then $xy = a^m a^n = a^{m+n} = a^{n+m}$ $a^n a^m = yx$, so G is abelian.

Problem .

Prove that a cyclic group with only one generator can have at most 2 elemer

Solution.

Let $B = \{e, a, a^2, a^3, ..., a^{n-1}\}$ be a cyclic group of n elements. Then a^{-1} a^{n-1} also generates G, because $(a^{-1})^i = (a^i)^{-1} = a^{n-i}$ for $i = 1, 2, ..., n-1$. Tl if G has only one generator, we must have $n - 1 = 1$ and $n = 2$. Of cour $G = \{e\}$ is also cyclic with one generator.

Problem

Show that a group with no proper nontrivial subgroups is cyclic.

Solution.

Let G be a group with no proper nontrivial subgroups. If $G = \{e\}$, then is of course cyclic. If $G \neq \{e\}$, then choose $a \in G$ such that $a \neq e$. We kn that $\langle a \rangle$ is a subgroup of G and $\langle a \rangle \neq \{e\}$. Because G has no proper nontriv subgroups, we must have $\langle a \rangle = G$. This shows that G is cyclic.

Problem

Let $\phi : G \to G'$ be an isomorphism of a group $\langle G, * \rangle$ with a group $\langle G',$ Then show that if G is cyclic, so is G'.

Solution.

Let a be a generator of G. We claim $\phi(a)$ is a generator of G'. Let $b' \in G'$. Because ϕ maps G onto G', there exists $b \in G$ such that $\phi(b) = b'$. Because a generates G, there exists $n \in \mathbb{Z}$ such that $b = a^n$. Because ϕ is an isomorphism, $b' = \phi(b) = \phi(a^n) = \phi(a)^n$. Thus G' is cyclic. ∎

Problemet

Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

Solution.

We have to prove that $\sim$ is reflexive, symmetric and transitive. Let $a \in G$. Then $aa^{-1} = e$ and $e \in H$, since H is a subgroup. Thus $a \sim a \implies \sim$ is reflexive. Let $a, b \in G$ and $a \sim b$, so that $ab^{-1} \in H$. Since H is a subgroup, we have $(ab^{-1})^{-1} = ba^{-1} \in H$, so $b \sim a \implies \sim$ is symmetric. Now, let $a, b, c \in G$ and $a \sim b$ and $b \sim c$. Then $ab^{-1} \in H$ and $bc^{-1} \in H$ so $(ab^{-1})(bc^{-1}) = ac^{-1} \in H$, implies $a \sim c$. Thus $\sim$ is transitive. ∎

Exercises.

Theorem

A subgroup of a cyclic group is cyclic.

Proof.

Let $G = \langle a \rangle$ and let $H \leq G$. If $H = \{e\}$, then $H = \langle e \rangle$ is cyclic. If $H \neq \langle e \rangle$,

then $a^n \in H$ for some $n \in \mathbb{Z}^+$. Choose m as the smallest integer in $\mathbb{Z}^+$ such that $a^m \in H$. We claim that $H = \langle a^m \rangle$. Let $b \in H$. Since $H \leq G$, we have $b = a^n$ for some $n \in \mathbb{Z}^+$. By division algorithm, there exists integers q and r such that $n = mq + r$ for $0 \leq r < m$. Then $a^n = a^{mq+r} = (a^m)^q a^r \implies a^r = (a^m)^{-q} a^n$. Since $a^n \in H, a^m \in H$, and H is a group, $(a^m)^{-q} \in H$, so that $(a^m)^{-q} a^n \in H \implies a^r \in H$. Since m was the smallest positive integer such that $a^m \in H$ and $0 \leq r < m$, we have $r = 0$. Thus $n = qm$, so that $b = a^n = (a^m)^q \implies H = \langle a^m \rangle$. Hence a subgroup of a cyclic group is cyclic.

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